Summary of previous work

My work so far has been centred around unordered configuration spaces on manifolds, in particular on questions of homological stability. If M is a connected, non-compact, smooth manifold, then the sequence of unordered configuration spaces $C_k(M)$ is known by results of McDuff and Segal [Seg73; McD75; Seg79] to be *homologically stable*, meaning that in a certain range of homological degrees $2i \leqslant k$, there are isomorphisms

$$H_i(C_k(M);\mathbb{Z}) \;\cong\; H_i(C_{k+1}(M);\mathbb{Z}).$$

These isomorphisms are moreover induced by *stabilisation maps* $C_k(M) \to C_{k+1}(M)$, which are defined roughly by choosing an end of the manifold M (possible since it is non-compact) and pulling in a new point "from infinity".

My work has focused on four related situations: (1) where unordered configuration spaces are replaced by certain double covers, (2) where one considers homology with twisted coefficients instead of constant $\mathbb{Z}$ coefficients, (3) where the manifold M is compact (in which case the question of homological stability is much more delicate, and not true in general with $\mathbb{Z}$ coefficients), and (4) where one considers configurations of embedded, disconnected submanifolds of higher dimension, instead of just points.

1. Oriented configuration spaces. Define the *oriented configuration space* $C_k^+(M)$ of k points in M to be the double cover of $C_k(M)$ whose fibre over a configuration $\{p_1,\dots,p_k\} \subset M$ is the set of orderings of the points $p_1,\dots,p_k$ modulo even permutations. My result concerning these spaces is that they also satisfy homological stability, with respect to analogous stabilisation maps, but only in the more restrictive range $3i+5 \leqslant k$. One may calculate the homology of some examples to see that the '3' in this range is optimal, and cannot be improved to a '2', in contrast the unordered configuration spaces. Reference: [Pal13a].

In joint work with Jeremy Miller [MP13], we identified the stable homology of oriented configuration spaces on M with a certain double cover of a section space of a bundle over M. This is analogous to the result of McDuff [McD75] for the unordered configuration spaces, using the *scanning map*, and indeed to prove our result we show that McDuff's scanning map is not just a homology equivalence, as she proved, but an *acyclic map* (a homology equivalence for all twisted coefficient systems) – a property which is inherited by double covers. To prove this we needed a version of the McDuff-Segal group-completion theorem [MS75] for homology with twisted coefficients, which we proved in [MP14].

2. Twisted homological stability. Returning to the unordered configuration spaces, we could choose a local coefficient system $\mathscr{L}_k$ for each $C_k(M)$, and ask whether homological stability holds for the sequence of groups $H_i(C_k(M);\mathscr{L}_k)$. Of course this cannot be true generally unless one imposes some relationships between the $\mathscr{L}_k$ for different k. This can be formalised by defining a certain category $\mathscr{C}$ whose objects are the natural numbers, and where the automorphism group $\mathrm{Aut}_{\mathscr{C}}(k)$ is isomorphic to $\pi_1(C_k(M))$ (note that $C_k(M)$ is path-connected as soon as M is at least 2-dimensional, since we are assuming that M is connected). A functor $\mathscr{L}\colon \mathscr{C} \to \mathsf{Ab}$ to the category of abelian groups encodes in particular the data of a local coefficient system $\mathscr{L}(k)$ for each space $C_k(M)$, and the additional morphisms $k \to \ell$ in $\mathscr{C}$ for $k \neq \ell$ may be used to impose certain finiteness conditions on the functor $\mathscr{L}$. In my preprint [Pal13b] I set up such a framework, and prove that if the functor $\mathscr{L}$ is *polynomial of degree* $\leqslant d$ for some finite d, then there are isomorphisms $H_i(C_k(M);\mathscr{L}(k)) \cong H_i(C_{k+1}(M);\mathscr{L}(k+1))$ in the range $2i+d \leqslant k$. This generalises results and techniques of Betley [Bet02], who considered the symmetric groups Σ_k (corresponding to the case $M=\mathbb{R}^\infty$).

3. Closed ambient manifolds. When M is closed, homological stability is not true in general, for example one may calculate that $H_1(C_k(S^2);\mathbb{Z}) \cong \mathbb{Z}/(2k-2)$, which does not stabilise as $k\to\infty$. Moreover, the stabilisation maps mentioned above do not exist, since M has nowhere 'at infinity' from which to pull in a new configuration point. In a joint project with Federico Cantero [CP14], we prove three main results which show that the homology of configuration spaces on closed manifolds exhibits a large amount of stability despite these issues.

(1) When the Euler characteristic χ of M is zero, we construct *replication maps* $C_k(M) \to C_{rk}(M)$ and show that these induce isomorphisms on $H_i(-,\mathbb{Z}[\frac{1}{r}])$ in the range $2i \leqslant k$.

(2) When the manifold is odd-dimensional, we show that there are isomorphisms

$$H_i(C_k(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{k+1}(M);\mathbb{Z}[\tfrac{1}{2}]) \qquad \text{and} \qquad H_i(C_k(M);\mathbb{Z}) \cong H_i(C_{k+2}(M);\mathbb{Z})$$

in the range $2i \leqslant k$, induced by a zigzag of maps. This strengthens a result of Bendersky-Miller [BM14].

(3) When the manifold is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many people [BCT89; ML88; Chu12; RW13; BM14; Knu14] that homological stability holds for $C_k(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_k(S^2);\mathbb{F}) \cong \begin{Bmatrix} \mathbb{F} & p \mid k-1 \\ 0 & p \nmid k-1 \end{Bmatrix} \quad \text{for } k \geqslant 2.$$

From this example we see that the first homology of $C_k(S^2)$ is not stable, but it is at least p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p. We then show that for each fixed i the sequence

$$H_i(C_k(M);\mathbb{F}) \quad \text{for } k \geqslant 2i$$

is p^{a+1}-periodic and takes on at most $a+2$ values. Moreover, if $\chi \equiv 1 \bmod p$ then the above sequence is 1-periodic, i.e. homological stability holds with coefficients in $\mathbb{F}$. The p^{a+1}-periodicity result is very similar to a theorem of Nagpal [Nag15], although his estimate of the period is much less explicit than ours.

4. Spaces of disconnected submanifolds. We now return to the case where M is non-compact, but instead of configuration spaces of points in M, we consider spaces of submanifolds of M which are diffeomorphic to k disjoint copies of a fixed closed manifold P. This is topologised as a path-component of the quotient space $\mathrm{Emb}(kP,M)/\mathrm{Diff}(kP)$, where kP denotes the disjoint union of k copies of P. In work in preparation [Pal15], I prove that these spaces are homologically stable with respect to k, just as for configurations of points, as long as $\dim(P) \leqslant \frac{1}{2}(\dim(M)-3)$.

Research interests

I am interested in many questions related to configuration spaces, and in particular the limiting behaviour of their homology. Some particular current and future projects are as follows.

Related to (2) above, I am working on generalising my twisted homological stability result to include additional interesting examples of local coefficient systems for configuration spaces $C_k(M)$. I would also like to compute the stable twisted homology of the $C_k(M)$ with coefficients in a polynomial functor $\mathscr{L}\colon \mathscr{C} \to \mathsf{Ab}$. For untwisted $\mathbb{Z}$ coefficients, this is known by [McD75] to be the homology of the space $\Gamma_c(\dot{T}M)_0$, where $\dot{T}M$ is the fibrewise one-point compactification of the tangent bundle of M, $\Gamma_c(-)$ denotes the space of sections of $\dot{T}M$ that agree with the ∞-section outside some compact subset, and the subscript $(-)_0$ means that we take just one path-component. However, the stable *twisted* homology of a sequence of spaces can be very different to the stable untwisted homology: for example, for the classifying spaces $B\mathrm{Aut}(F_n)$ of automorphism groups of free groups, the latter is the homology of one path-component of $\Omega^\infty S^\infty$, by [Gal11], whereas the former, for many twisted coefficient systems, is trivial, by [DV12].

Related to (3) above, Federico Cantero and I are continuing our project by studying the algebraic structure induced by the replication maps (and generalisations thereof) on the homology of configuration spaces. I am also interested in combining the replication map with ideas from (2) above to study the twisted homology of configuration spaces on closed manifolds.

Related to (4) above, I would like to improve the restriction on the relative dimensions of P and M to the weaker assumption that $\dim(P) \leqslant \frac{1}{2}(\dim(M)-1)$, which would include the case of links in 3-manifolds. I am also working on applying the result of homological stability for spaces of disconnected submanifolds to obtain a homological stability result for certain diffeomorphism groups of sequences of manifolds. Finally, I am interested in the question of the stable homology of spaces of disconnected submanifolds of M. Unlike in the case of configurations of points, or spaces of connected subsurfaces [CRW13], the naturally-defined scanning map is *not* in general a homology isomorphism in the limit, so some new approach will be needed for this question.

References

[BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123.

[Bet02] Stanislaw Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic).

[BM14] Martin Bendersky and Jeremy Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815. {arxiv:1212.3596}.

[Chu12] Thomas Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504. {arxiv:1103.2441}.

[CP14] Federico Cantero and Martin Palmer. *On homological stability for configuration spaces on closed background manifolds*. ArXiv:1406.4916v2. 2014. To appear in Documenta Mathematica.

[CRW13] Federico Cantero and Oscar Randal-Williams. *Homological stability for spaces of surfaces*. ArXiv:1304.3006v2. 2013 (v2: 2014).

[DV12] Aurélien Djament and Christine Vespa. *Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux*. ArXiv:1210.4030v3. 2012 (v3: 2013). To appear in Comment. Math. Helv.

[Gal11] Søren Galatius. *Stable homology of automorphism groups of free groups*. *Ann. of Math. (2)* 173.2 (2011), pp. 705–768. {arxiv:math/0610216}.

[Knu14] Ben Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. ArXiv:1405.6696v4. 2014.

[McD75] Dusa McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107.

[ML88] R. James Milgram and Peter Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424.

[MP13] Jeremy Miller and Martin Palmer. *Scanning for oriented configuration spaces*. ArXiv:1306.6896v2. 2013 (v2: 2014). To appear in HHA.

[MP14] Jeremy Miller and Martin Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. ArXiv:1409.4389v1. 2014. To appear in Q. J. Math.

[MS75] D. McDuff and G. Segal. *Homology fibrations and the "group-completion" theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284.

[Nag15] Rohit Nagpal. *FI-modules and the cohomology of modular S_n representations*. PhD thesis. University of Wisconsin-Madison, 2015.

[Pal13a] Martin Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711. {arxiv:1106.4540}.

[Pal13b] Martin Palmer. *Twisted homological stability for configuration spaces*. ArXiv:1308.4397v2. 2013 (v2: 2014).

[Pal15] Martin Palmer. *Homological stability for spaces of disconnected submanifolds*. In preparation. 2015.

[RW13] Oscar Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326. {arxiv:1105.5257}.

[Seg73] Graeme Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221.

[Seg79] Graeme Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72.